Chapter eight

Frequency Characteristics of AC Circuits

Objectives

When you have studied the material in this chapter, you should be able to:

- **design simple high-pass and low-pass networks using *RC* or *RL* circuits**
- **explain how the gain and phase shift produced by these circuits varies with frequency**
- **predict the effects of combining a number of high-pass or low-pass stages and outline the characteristics of the resulting arrangement**
- **describe the characteristics of simple circuits containing resistors, inductors and capacitors, and calculate the resonant frequency and bandwidth of such circuits**
- **discuss the operation and characteristics of a range of passive and active filters**
- **explain the importance of stray capacitance and stray inductance in determining the frequency characteristics of electronic circuits.**

8.1 Introduction

Having studied the AC behaviour of some basic circuit components, we are now in a position to consider their effects on the frequency characteristics of simple circuits.

While the properties of a pure resistance are not affected by the frequency of the signal concerned, this is not true of reactive components. The reactance of both inductors and capacitors is dependent on frequency, and therefore the characteristics of any circuit that includes capacitors or inductors will change with frequency. However, the situation is more complex than this because, as we noted in Chapters 4 and 5, all real circuits have both stray capacitance and stray inductance. Inevitably, therefore, the characteristics of all circuits will change with frequency.

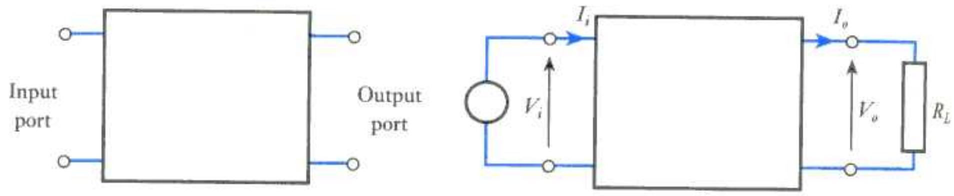
In order to understand the nature of these frequency-related effects we will look at simple combinations of resistors, capacitors and inductors and see how their characteristics change with frequency. However, before looking at these circuits it is useful to introduce a couple of new concepts and techniques.

8.2 Two-port networks

A two-port network is, as its name suggests, simply a circuit configuration that has two 'ports', namely the input port and the output port. Such an arrangement is shown in Figure 8.1(a).

When connected to some input circuitry (perhaps a voltage source) and to some output circuitry (such as a load resistor) we can then identify the

Figure 8.1 A two-port network.



(a) A two-port network (b) A typical arrangement

voltages at the input and the output (V_i and V_o) and the currents flowing into and out of the network (I_i and I_o). Such an arrangement is illustrated in Figure 8.1(b). Clearly the relationship between the output voltage and the output current is determined by the value of the *load resistance* R_L. Similarly, the relationship between the input voltage and the input current determines the effective resistance looking into the input port of the arrangement. This is termed the **input resistance** of the network, which is given the symbol R_i and clearly is equal to V_i/I_i. We can then use these various voltages and currents to describe the characteristics of the two-port network.

The ratio of the output voltage to the input voltage is termed the **voltage gain** of the circuit, while the ratio of the output current to the input current is termed the **current gain**. The **power gain** of the network is the ratio of the power supplied to a load to the power absorbed from the source. The input power can be calculated from the input voltage and the input current, and the output power can be determined from the output voltage and output current. Thus

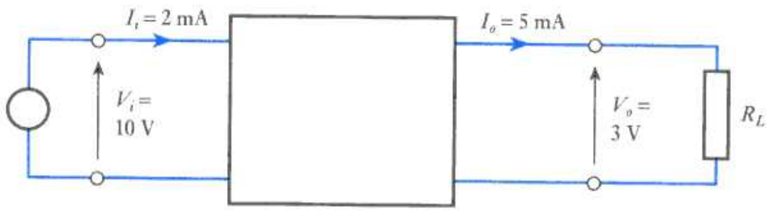
$$\text{voltage gain } (A_v) = \frac{V_o}{V_i} \tag{8.1}$$

$$\text{current gain } (A_i) = \frac{I_o}{I_i} \tag{8.2}$$

$$\text{power gain } (A_p) = \frac{P_o}{P_i} \tag{8.3}$$

Example 8.1

Calculate the voltage, current gain and power gain of the following two-port network.



From the diagram, and from Equations 8.1 to 8.3 above, we see that

$$\text{voltage gain } (A_v) = \frac{V_o}{V_i} = \frac{3\text{ V}}{10\text{ V}} = 0.3$$

$$\text{current gain } (A_i) = \frac{I_o}{I_i} = \frac{5\text{ mA}}{2\text{ mA}} = 2.5$$

$$\text{power gain } (A_p) = \frac{P_o}{P_i} = \frac{V_o \times I_o}{V_i \times I_i} = \frac{3\text{ V} \times 5\text{ mA}}{10\text{ V} \times 2\text{ mA}} = 0.75$$

Note that the various gains can each be greater or less than unity. A gain of greater than unity represents *amplification* while a gain of less than unity represents *attenuation*. A power gain of greater than unity implies that the circuit is delivering more power to the load than it is accepting at its input. Such an arrangement requires some form of external power source. *Passive* circuits, such as combinations of resistors, capacitors and inductors, will always have a power gain that is no greater than unity. *Active* circuits, which use an external power supply, can have a power gain that is very much greater than unity.

The power gain of a modern electronic amplifier may be very high, gains of 10^6 or 10^7 being common. With these large numbers it is often convenient to use a logarithmic expression of gain rather than a simple ratio. This is often done using **decibels**.

8.3 The decibel (dB)

The decibel (dB) is a dimensionless figure for **power gain** and is defined by

$$\text{power gain (dB)} = 10 \log_{10} \frac{P_2}{P_1} \tag{8.4}$$

where P_2 is the output power and P_1 is the input power of the amplifier or other circuit.

Example 8.2 Express a power gain of 2500 in decibels.

$$\begin{aligned}\text{power gain (dB)} &= 10 \log_{10} \frac{P_2}{P_1}\\ &= 10 \log_{10} 2500\\ &= 10 \times 3.40\\ &= 34.0 \text{ dB}\end{aligned}$$

Decibels may be used to represent both amplification and attenuation, and, in addition to making large numbers more manageable, the use of decibels has several other advantages. For example, when several stages of amplification or attenuation are connected in series (this is often referred to as **cascading** circuits), the overall gain of the combination can be found simply by adding the individual gains of each stage when these are expressed in decibels. This is illustrated in Figure 8.2. The use of decibels also simplifies the description of the frequency response of circuits, as we will see later in this chapter.

For certain values of gain, the decibel equivalents are easy to remember or to calculate using mental arithmetic. Since $\log_{10} n$ is simply the power to which 10 must be raised to equal n, for powers of 10 it is easy to calculate. For example, $\log_{10} 10 = 1$, $\log_{10} 100 = 2$, $\log_{10} 1000 = 3$, and so on. Similarly, $\log_{10} 1/10 = -1$, $\log_{10} 1/100 = -2$ and $\log_{10} 1/1000 = -3$. Therefore, gains of

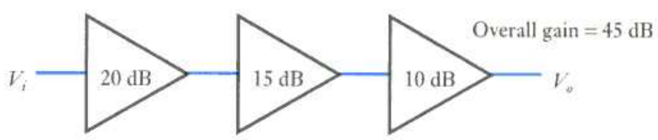


Figure 8.2 Calculating the gain of several stages in series.

Table 8.1 Expressing power amplification and attenuation in decibels.

Power gain (ratio)	Decibels (dB)
1000	30
100	20
10	10
2	3
1	0
0.5	−3
0.1	−10
0.01	−20
0.001	−30

10, 100 and 1000 are simply 10 dB, 20 dB and 30 dB respectively, and attenuations of 1/10, 1/100 and 1/1000 are simply −10 dB, −20 dB and −30 dB. A circuit that doubles the power has a gain of +3 dB, while a circuit that halves the power has a gain of −3 dB. A circuit that leaves the power unchanged (a power gain of 1) has a gain of 0 dB. These results are summarised in Table 8.1.

In many cases, our knowledge of a circuit relates to its voltage gain rather than to its power gain. Clearly, these two measures are related, and we know that the power dissipated in a resistance R is related to the applied voltage V by the expression V^2/R. Therefore, the gain of an amplifier expressed in decibels can be written as

$$\text{power gain (dB)} = 10\log_{10}\frac{P_2}{P_1} = 10\log_{10}\frac{V_2^2/R_2}{V_1^2/R_1}$$

where V_1 and V_2 are the input and output voltages, respectively, and R_1 and R_2 are the input and load resistances, respectively.

If, and only if, R_1 and R_2 are equal, the power gain of the amplifier is given by

$$\begin{aligned}\text{power gain (dB)} &= 10\log_{10}\frac{V_2^2}{V_1^2}\\ &= 20\log_{10}\frac{V_2}{V_1}\\ &= 20\log_{10}(\text{voltage gain})\end{aligned}$$

Some networks do have equal input and load resistance, and in these cases it is often useful to express the gain in decibels rather than as a simple ratio. Note that it is not strictly correct to say, for example, that a circuit has a voltage gain of 10 dB, even though you will often hear such statements. Decibels represent power gain, and what is meant is that the circuit has a voltage gain that corresponds to a power gain of 10 dB. However, it is very common to describe the voltage gain of a circuit in dB as

$$\text{voltage gain (dB)} = 20\log_{10}\frac{V_2}{V_1} \tag{8.5}$$

even when R_1 and R_2 are not equal.

Example 8.3

Calculate the gain in decibels of circuits that have power gains of 5, 50 and 500 and voltage gains of 5, 50 and 500.

Power gain of 5	Gain (dB) = $10 \log_{10}(5)$	7.0 dB
Power gain of 50	Gain (dB) = $10 \log_{10}(50)$	17.0 dB
Power gain of 500	Gain (dB) = $10 \log_{10}(500)$	27.0 dB
Voltage gain of 5	Gain (dB) = $20 \log_{10}(5)$	14.0 dB
Voltage gain of 50	Gain (dB) = $20 \log_{10}(50)$	34.0 dB
Voltage gain of 500	Gain (dB) = $20 \log_{10}(500)$	54.0 dB

Converting from gains expressed in decibels to simple power or voltage ratios requires the reversal of the operations used above. For example, since

$$\text{power gain (dB)} = 10 \log_{10}(\text{power gain})$$

it follows that

$$10 \log_{10}(\text{power gain}) = \text{power gain (dB)}$$

$$\log_{10}(\text{power gain}) = \frac{\text{power gain (dB)}}{10}$$

$$\text{power gain} = 10^{(\text{power gain (dB)}/10)} \tag{8.6}$$

Similarly,

$$\text{voltage gain} = 10^{(\text{power gain (dB)}/20)} \tag{8.7}$$

Example 8.4

Express gains of 20 dB, 30 dB and 40 dB as both power gains and voltage gains.

20 dB	$20 = 10 \log_{10}(\text{power gain})$ power gain = 10^2	power gain = 100
	$20 = 20 \log_{10}(\text{voltage gain})$ power gain = 10	voltage gain = 10
30 dB	$30 = 10 \log_{10}(\text{power gain})$ power gain = 10^3	power gain = 1000
	$30 = 20 \log_{10}(\text{voltage gain})$ power gain = $10^{1.5}$	voltage gain = 31.6
40 dB	$40 = 10 \log_{10}(\text{power gain})$ power gain = 10^4	power gain = 10,000
	$40 = 20 \log_{10}(\text{voltage gain})$ power gain = 10^2	voltage gain = 100

8.4 Frequency response

Since the characteristics of reactive components change with frequency, the behaviour of circuits using these components will also change. The way in which the gain of a circuit changes with frequency is termed its **frequency response**. These changes take the form of variations in the magnitude of the gain and in its phase angle, leading to two aspects of the response, namely the *amplitude response* and the *phase response*. In some situations both aspects are of importance, while in others only the amplitude response is needed. For this reason, the term *frequency response* is often used to refer simply to the amplitude response of a system.

In order to understand the nature of these frequency-related effects, we will start by looking at very simple circuits containing resistors and capacitors, or resistors and inductors. In Chapter 6, we looked at circuits involving impedances, including the potential divider arrangement shown in Figure 8.3. From our earlier consideration of the circuit, we know that the output voltage of this circuit is given by

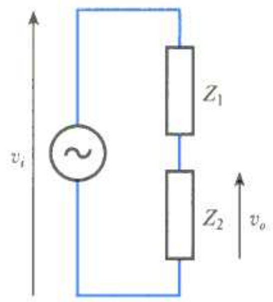


Figure 8.3 A potential divider circuit.

$$v_o = v_i \times \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

Another way of describing the behaviour of this circuit is to give an expression for the output voltage divided by the input voltage. In this case, this gives

$$\frac{v_o}{v_i} = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} \tag{8.8}$$

This ratio is the **voltage gain** of the circuit, but it is also referred to as its **transfer function**. We will now use this expression to analyse the behaviour of simple *RC* and *RL* circuits.

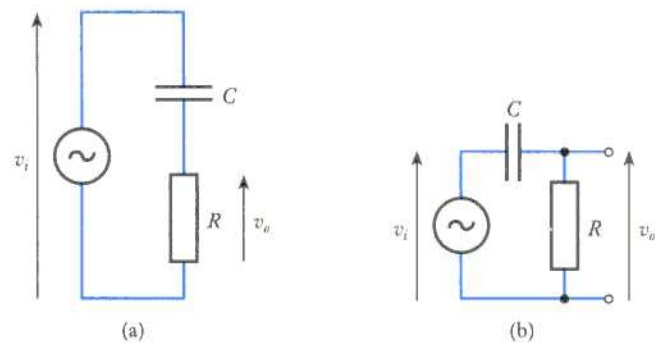
8.5 A high-pass *RC* network

Consider the circuit of Figure 8.4(a), which shows a potential divider, formed from a capacitor and a resistor. This circuit is shown redrawn in Figure 8.4(b), which is electrically identical. Applying Equation 8.8, we see that

$$\frac{v_o}{v_i} = \frac{\mathbf{Z}_R}{\mathbf{Z}_R + \mathbf{Z}_C} = \frac{R}{R - \mathrm{j}\dfrac{1}{\omega C}} = \frac{1}{1 - \mathrm{j}\dfrac{1}{\omega CR}} \tag{8.9}$$

At high frequencies, ω is large and the value of $1/\mathrm{j}\omega CR$ is small compared with 1. Therefore, the denominator of the expression is close to unity and the voltage gain is approximately 1.

Figure 8.4 A simple *RC* network.



However, at lower frequencies the magnitude of $1/\omega CR$ becomes more significant and the gain of the network decreases. Since the denominator of the expression for the gain has both real and imaginary parts, the magnitude of the voltage gain is given by

$$|\text{voltage gain}| = \frac{1}{\sqrt{1^2 + \left(\dfrac{1}{\omega CR}\right)^2}}$$

When the value of $1/\omega CR$ is equal to 1, this gives

$$|\text{voltage gain}| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$$

Since power gain is proportional to the square of the voltage gain, this is a halving of the power gain (or a fall of 3 dB) compared with the gain at high frequencies. This is termed the **cut-off frequency** of the circuit. If the angular frequency corresponding to this cut-off frequency is given the symbol ω_c, then $1/\omega_c CR$ is equal to 1, and

$$\omega_c = \frac{1}{CR} = \frac{1}{\mathrm{T}} \text{ rad/s} \qquad (8.10)$$

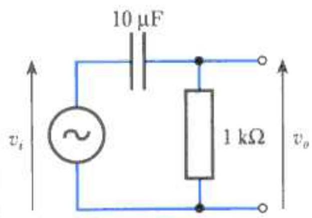
where $\mathrm{T} = CR$ is the time constant of the capacitor–resistor combination that produces the cut-off frequency.

Since it is often more convenient to deal with *cyclic* frequencies (which are measured in hertz) rather than *angular* frequencies (which are measured in radians per second) we can use the relationship $\omega = 2\pi f$ to calculate the corresponding cyclic cut-off frequency f_c:

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi CR} \text{Hz} \qquad (8.11)$$

Example 8.5

Calculate the time constant T, the angular cut-off frequency ω_c and the cyclic cut-off frequency f_c of the following arrangement.



From above

$$\mathrm{T} = CR = 10 \times 10^{-6} \times 1 \times 10^3 = 0.01 \text{ s}$$

$$\omega_c = \frac{1}{\mathrm{T}} = \frac{1}{0.01} = 100 \text{ rad/s}$$

$$f_c = \frac{\omega_c}{2\pi} = \frac{100}{2\pi} = 15.9 \text{ Hz}$$

If we substitute for ω (where $\omega = 2\pi f$) and CR (where $CR = 1/2\pi f_c$) in Equation 8.9, we obtain an expression for the gain of the circuit in terms of the signal frequency f and the cut-off frequency f_c:

$$\frac{v_o}{v_i} = \frac{1}{1 - j\frac{1}{\omega CR}} = \frac{1}{1 - j\frac{1}{(2\pi f)(1/2\pi f_c)}} = \frac{1}{1 - j\frac{f_c}{f}} \tag{8.12}$$

This is a general expression for the voltage gain of this form of CR network.

From Equation 8.12, it is clear that the voltage gain is a function of the signal frequency f and that the magnitude of the gain varies with frequency. Since the gain has an imaginary component, it is also clear that the circuit produces a **phase shift** that changes with frequency. To investigate how these two quantities change with frequency, let us consider the gain of the circuit in different frequency ranges.

8.5.1 When $f \gg f_c$

When the signal frequency f is much greater than the cut-off frequency f_c, then in Equation 8.12 f_c/f is much less than unity, and the voltage gain is approximately equal to 1. Here the imaginary part of the gain is negligible and the gain of the circuit is effectively real. Hence the phase shift produced is negligible. This situation is shown in the phasor diagram of Figure 8.5(a).

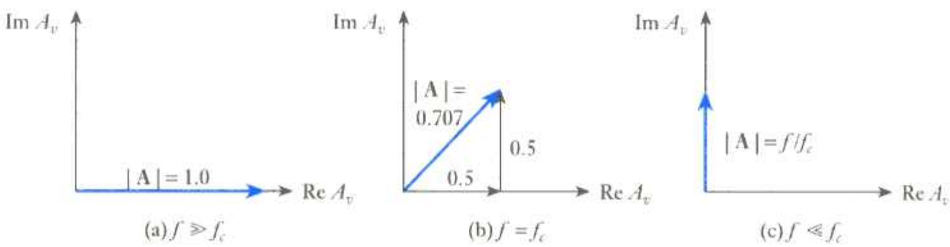


Figure 8.5 Phasor diagrams of the gain of the circuit of Figure 8.4 at different frequencies.

8.5.2 When $f = f_c$

When the signal frequency f is equal to the cut-off frequency f_c, then Equation 8.12 becomes

$$\frac{v_o}{v_i} = \frac{1}{1 - j\frac{f_c}{f}} = \frac{1}{1 - j}$$

Multiplying the numerator and the denominator by $(1 + j)$ gives

$$\frac{v_o}{v_i} = \frac{(1+j)}{(1-j)(1+j)} = \frac{(1+j)}{2} = 0.5 + 0.5j$$

This is illustrated in the phasor diagram of Figure 8.5(b), which shows that the magnitude of the gain at the cut-off frequency is 0.707. This is consistent with our earlier analysis, which predicted that the gain at the cut-off frequency should be $1/\sqrt{2}$ (or 0.707) times the mid-band gain. In this case, the mid-band gain is the gain some way above the cut-off frequency, which we have just shown to be 1. The phasor diagram also shows that at this frequency the phase angle of the gain is +45°. This shows that the output voltage *leads* the input voltage by 45°. The gain is therefore $0.707\angle 45°$.

8.5.3 When $f \ll f_c$

The third region of interest is where the signal frequency is well below the cut-off frequency. Here f_c/f is much greater than 1, and Equation 8.12 becomes

$$\frac{v_o}{v_i} = \frac{1}{1 - j\dfrac{f_c}{f}} \approx \frac{1}{-j\dfrac{f_c}{f}} = j\frac{f}{f_c}$$

The 'j' signifies that the gain is imaginary, as shown in the phasor diagram of Figure 8.5(c). The magnitude of the gain is simply f/f_c and the phase shift is +90°, the '+' sign meaning that the output voltage *leads* the input voltage by 90°.

Since f_c is a constant for a given circuit, in this region the voltage gain is linearly related to frequency. If the frequency is halved the voltage gain will be halved. Therefore, the gain falls by a factor of 0.5 for every octave drop in frequency (an **octave** is a doubling or halving of frequency and is equivalent to an octave jump on a piano or other musical instrument). A fall in voltage gain by a factor of 0.5 is equivalent to a change in gain of −6 dB. Therefore, the rate of change of gain can be expressed as 6 dB per octave. An alternative way of expressing the rate of change of gain is to specify the change of gain for a decade change in frequency (a **decade**, as its name suggests, is a change in frequency of a factor of 10). If the frequency falls to 0.1 of its previous value, the voltage gain will also drop to 0.1 of its previous value. This represents a change in gain of −20 dB. Thus the rate of change of gain is 20 dB per decade.

Example 8.6

Determine the frequencies corresponding to:

(a) an octave above 1 kHz;
(b) three octaves above 10 Hz;
(c) an octave below 100 Hz;
(d) a decade above 20 Hz;
(e) three decades below 1 MHz;
(f) two decades above 50 Hz.

(a) an octave above 1 kHz = 1000 × 2 = 2 kHz
(b) three octaves above 10 Hz = 10 × 2 × 2 × 2 = 80 Hz
(c) an octave below 100 Hz = 100 ÷ 2 = 50 Hz
(d) a decade above 20 Hz = 20 × 10 = 200 Hz
(e) three decades below 1 MHz = 1,000,000 ÷ 10 ÷ 10 ÷ 10 = 1 kHz
(f) two decades above 50 Hz = 50 × 10 × 10 = 5 kHz

8.5.4 Frequency response of the high-pass *RC* network

Figure 8.6 shows the gain and phase response of the circuit of Figure 8.4 for frequencies above and below the cut-off frequency. It can be seen that, at

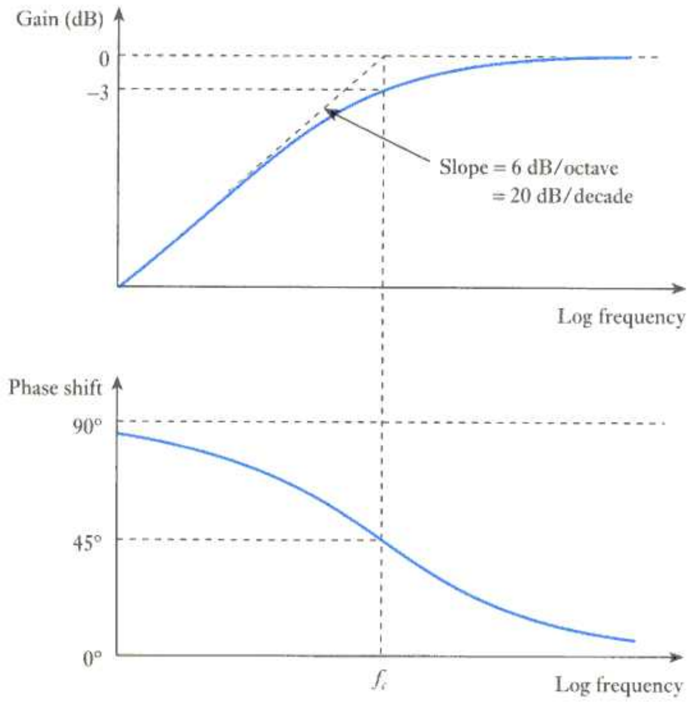


Figure 8.6 Gain and phase responses (or Bode diagram) for the high-pass *RC* network.

frequencies much greater than the cut-off frequency, the magnitude of the gain tends to a straight line corresponding to a gain of 0 dB (that is, a gain of 1). Therefore, this line (shown dashed in Figure 8.6) forms an **asymptote** to the response. At frequencies much less than the cut-off frequency, the response tends to a straight line drawn at a slope of 6 dB per octave (20 dB per decade) change in frequency. This line forms a second asymptote to the response and is also shown dashed on Figure 8.6. The two asymptotes intersect at the cut-off frequency. At frequencies considerably above or below the cut-off frequency, the gain response tends towards these two asymptotes. Near the cut-off frequency, the gain deviates from the two straight lines and is 3 dB below their intersection at the cut-off frequency.

Figure 8.6 also shows the variation of phase with frequency of the *RC* network. At frequencies well above the cut-off frequency, the network produces very little phase shift and its effects may generally be ignored. However, as the frequency decreases the phase shift produced by the arrangement increases, reaching 45° at the cut-off frequency and increasing to 90° at very low frequencies.

Asymptotic diagrams of gain and phase of the form shown in Figure 8.6 are referred to as **Bode diagrams** (or sometimes **Bode plots**). These plot logarithmic gain (usually in dB) and phase against logarithmic frequency. Such diagrams are easy to plot and give a useful picture of the characteristic of the circuit. We will look at the Bode diagrams for a range of other circuits in this chapter and then consider how they may be easily drawn and used.

It can be seen that the *RC* network passes signals of some frequencies with little effect but that signals of other frequencies are attenuated and are subjected to a phase shift. The network therefore has the characteristics of a **high-pass filter**, since it allows high-frequency signals to pass but filters out low-frequency signals. We will look at filters in more detail later in this chapter.

File 8A

Computer simulation exercise 8.1

Calculate the cut-off frequency of the circuit of Figure 8.4 if $R = 1\ \text{k}\Omega$ and $C = 1\ \mu\text{F}$. Simulate the circuit using these component values and perform an AC sweep to measure the response over a range from 1 Hz to 1 MHz. Plot the gain (in dB) and the phase of the output over this frequency range, estimate the cut-off frequency from these plots and compare this with the predicted value. Measure the phase shift at the estimated cut-off frequency and compare this with the value predicted above. Repeat this exercise for different values of R and C.

8.6 A low-pass *RC* network

The circuit of Figure 8.7 shows an *RC* arrangement similar to the earlier circuit but with the positions of the resistor and the capacitor reversed. Applying Equation 8.8 produces

$$\frac{v_o}{v_i} = \frac{\mathbf{Z}_C}{\mathbf{Z}_R + \mathbf{Z}_C} = \frac{-j\dfrac{1}{\omega C}}{R - j\dfrac{1}{\omega C}} = \frac{1}{1 + j\omega CR} \tag{8.13}$$

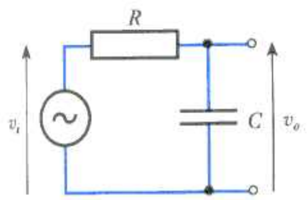


Figure 8.7 A low-pass *RC* network.

Comparing this expression with that of Equation 8.9 shows that it has a very different frequency characteristic. At low frequencies, ω is small and the value of $j\omega CR$ is small compared with 1. Therefore, the denominator of the expression is close to unity and the voltage gain is approximately 1. At high frequencies, the magnitude of ωCR becomes more significant and the gain of the network decreases. We therefore have a **low-pass filter** arrangement.

A similar analysis to that in the last section will show that the magnitude of the voltage gain is now given by

$$|\,\text{voltage gain}\,| = \frac{1}{\sqrt{1 + (\omega CR)^2}}$$

When the value of ωCR is equal to 1, this gives

$$|\,\text{voltage gain}\,| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$$

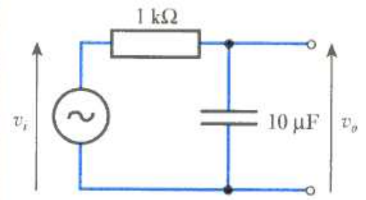
and again this corresponds to a cut-off frequency. The angular frequency of the cut-off ω_c corresponds to the condition that $\omega CR = 1$, therefore

$$\omega_c = \frac{1}{CR} = \frac{1}{\text{T}}\ \text{rad/s} \tag{8.14}$$

as before. Therefore the expression for the cut-off frequency is identical to that in the previous circuit.

Example 8.7

Calculate the time constant T, the angular cut-off frequency ω_c and the cyclic cut-off frequency f_c of the following arrangement.



From above

$$T = CR = 10 \times 10^{-6} \times 1 \times 10^{3} = 0.01 \text{ s}$$

$$\omega_c = \frac{1}{T} = \frac{1}{0.01} = 100 \text{ rad/s}$$

$$f_c = \frac{\omega_c}{2\pi} = \frac{100}{2\pi} = 15.9 \text{ Hz}$$

While the cut-off frequency of this circuit is identical to that of the previous arrangement, you should note that in the circuit of Figure 8.4 the cut-off attenuates low-frequency signals and is therefore a **low-frequency cut-off**. However, in the circuit of Figure 8.7 high frequencies are attenuated, so this circuit has a **high-frequency cut-off**.

Substituting into Equation 8.13 gives

$$\frac{v_o}{v_i} = \frac{1}{1 + j\omega CR} = \frac{1}{1 + j\frac{\omega}{\omega_c}} = \frac{1}{1 + j\frac{f}{f_c}} \tag{8.15}$$

You might like to compare this with the expression for a high-pass network in Equation 8.12. As before, we can investigate the behaviour of this arrangement in different frequency ranges.

8.6.1 When $f \ll f_c$

When the signal frequency f is much lower than the cut-off frequency f_c, then in Equation 8.15 f/f_c is much less than unity, and the voltage gain is approximately equal to 1. The imaginary part of the gain is negligible and the gain of the circuit is effectively real. This situation is shown in the phasor diagram of Figure 8.8(a).

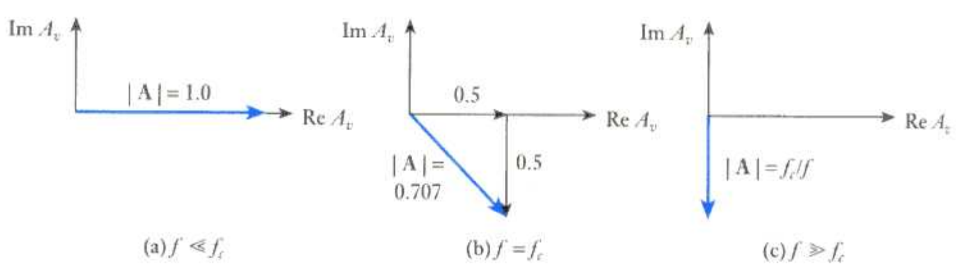


Figure 8.8 Phasor diagrams of the gain of the low-pass network at different frequencies.

8.6.2 When $f = f_c$

When the signal frequency f is equal to the cut-off frequency f_c, then Equation 8.15 becomes

$$\frac{v_o}{v_i} = \frac{1}{1 + j\dfrac{f}{f_c}} = \frac{1}{1 + j}$$

Multiplying the numerator and the denominator by $(1 - j)$ gives

$$\frac{v_o}{v_i} = \frac{(1 - j)}{(1 + j)(1 - j)} = \frac{(1 - j)}{2} = 0.5 - 0.5j$$

This is illustrated in the phasor diagram of Figure 8.8(b), which shows that the magnitude of the gain at the cut-off frequency is 0.707 and the phase angle of the gain is $-45°$. This shows that the output voltage *lags* the input voltage by 45°. The gain is therefore $0.707\angle{-45°}$.

8.6.3 When $f \gg f_c$

At high frequencies f/f_c is much greater than 1, and Equation 8.15 becomes

$$\frac{v_o}{v_i} = \frac{1}{1 + j\dfrac{f}{f_c}} \approx \frac{1}{j\dfrac{f}{f_c}} = -j\frac{f_c}{f}$$

The 'j' signifies that the gain is imaginary, and the minus sign indicates that the output lags the input. This is shown in the phasor diagram of Figure 8.8(c). The magnitude of the gain is simply f_c/f and, since f_c is a constant, the voltage gain is inversely proportional to frequency. If the frequency is halved, the voltage gain will be doubled. Therefore, the rate of change of gain can be expressed as −6 dB/octave or −20 dB/decade.

8.6.4 Frequency response of the low-pass *RC* network

Figure 8.9 shows the gain and phase response (or Bode diagram) of the low-pass network for frequencies above and below the cut-off frequency. The magnitude response is very similar in form to that of the high-pass network shown in Figure 8.6, with the frequency scale reversed. The phase response is a similar shape to that in Figure 8.6, but here the phase goes from 0° to −90° as the frequency is increased, rather than from +90° to 0° as in the previous arrangement. From the figure it is clear that this is a low-pass filter arrangement.

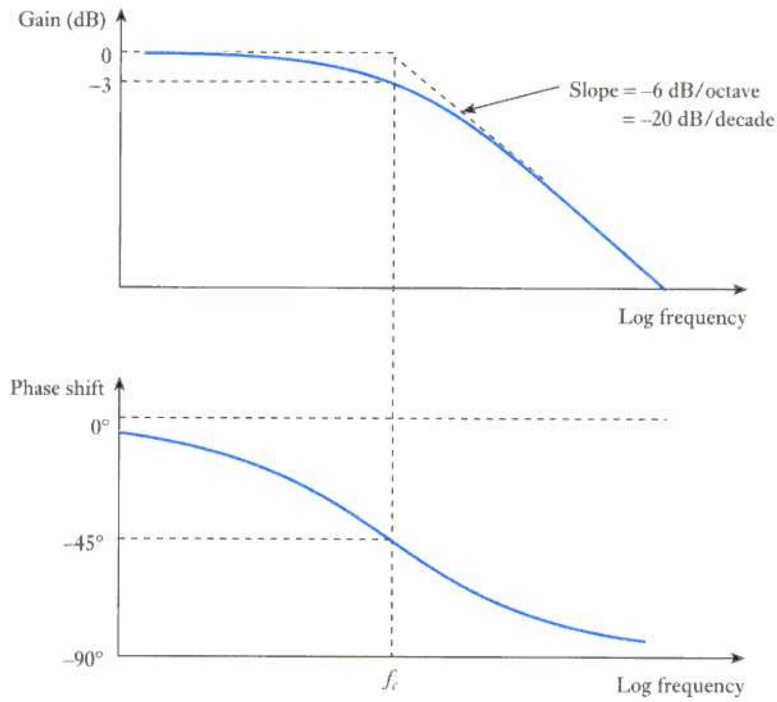


Figure 8.9 Gain and phase responses (or Bode diagram) for the low-pass *RC* network.

File 8B

Computer simulation exercise 8.2

Calculate the cut-off frequency of the circuit of Figure 8.7 if $R = 1\ \mathrm{k\Omega}$ and $C = 1\ \mu\mathrm{F}$. Simulate the circuit using these component values and perform an AC sweep to measure the response over a range from 1 Hz to 1 MHz. Plot the gain (in dB) and the phase of the output over this frequency range, estimate the cut-off frequency from these plots and compare this with the predicted value. Measure the phase shift at the estimated cut-off frequency and compare this with the value predicted above. Repeat this exercise for different values of *R* and *C*.

8.7 A low-pass *RL* network

High-pass and low-pass arrangements may also be formed using combinations of resistors and inductors. Consider for example the circuit of Figure 8.10. This shows a circuit similar to that of Figure 8.4, but with the capacitor replaced by an inductor. If we apply a similar analysis to that used above, we obtain

$$\frac{v_o}{v_i} = \frac{\mathbf{Z}_R}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{R}{R + \mathrm{j}\omega L} = \frac{1}{1 + \mathrm{j}\omega \dfrac{L}{R}} \tag{8.16}$$

A similar analysis to that in the last section will show that the magnitude of the voltage gain is now given by

$$|\,\text{voltage gain}\,| = \frac{1}{\sqrt{1 + \left(\omega \dfrac{L}{R}\right)^2}}$$

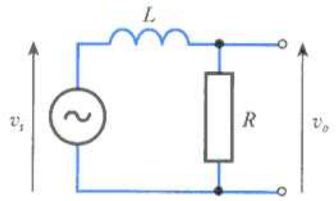


Figure 8.10 A low-pass *RL* network.

When the value of $\omega L/R$ is equal to 1, this gives

$$|\text{voltage gain}| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$$

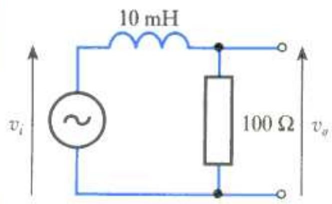
and this corresponds to a cut-off frequency. The angular frequency of the cut-off ω_c corresponds to the condition that $\omega L/R = 1$, therefore

$$\omega_c = \frac{R}{L} = \frac{1}{\mathsf{T}} \text{ rad/s} \tag{8.17}$$

As before, T is the time constant of the circuit, and in this case T is equal to L/R.

Example 8.8

Calculate the time constant T, the angular cut-off frequency ω_c and the cyclic cut-off frequency f_c of the following arrangement.



From above

$$\mathsf{T} = \frac{L}{R} = \frac{10 \times 10^{-3}}{100} = 10^{-4} \text{ s}$$

$$\omega_c = \frac{1}{\mathsf{T}} = \frac{1}{1 \times 10^{-4}} = 10^4 \text{ rad/s}$$

$$f_c = \frac{\omega_c}{2\pi} = \frac{1 \times 10^4}{2\pi} = 1.59 \text{ kHz}$$

Substituting into Equation 8.16, we have

$$\frac{v_o}{v_i} = \frac{1}{1 + j\omega \frac{L}{R}} = \frac{1}{1 + j\frac{\omega}{\omega_c}} = \frac{1}{1 + j\frac{f}{f_c}} \tag{8.18}$$

This expression is identical to that of Equation 8.15, and thus the frequency behaviour of this circuit is identical to that of the circuit of Figure 8.7.

File 8C

Computer simulation exercise 8.3

Calculate the cut-off frequency of the circuit of Figure 8.10 if $R = 10\ \Omega$ and $L = 5$ mH. Simulate the circuit using these component values and perform an AC sweep to measure the response over a range from 1 Hz to 1 MHz. Plot the gain (in dB) and the phase of the output over this frequency range, estimate the cut-off frequency from these plots and compare this with the predicted value. Measure the phase shift at the estimated cut-off frequency and compare this with the value predicted above. Repeat this exercise for different values of R and L.

8.8 A high-pass *RL* network

Interchanging the components of Figure 8.10 gives the circuit of Figure 8.11. Analysing this as before, we obtain

$$\frac{v_o}{v_i} = \frac{\mathbf{Z}_L}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{j\omega L}{R + j\omega L} = \frac{1}{1 + \dfrac{R}{j\omega L}} = \frac{1}{1 - j\dfrac{R}{\omega L}} \tag{8.19}$$

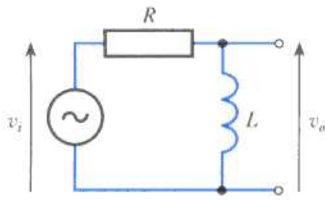


Figure 8.11 A high-pass *RL* network.

If we substitute $\omega_c = R/L$ as before, this gives

$$\frac{v_o}{v_i} = \frac{1}{1 - j\dfrac{R}{\omega L}} = \frac{1}{1 - j\dfrac{\omega_c}{\omega}} = \frac{1}{1 - j\dfrac{f_c}{f}} \tag{8.20}$$

This expression is identical to that of Equation 8.12, and thus the frequency behaviour of this circuit is identical to that of the circuit of Figure 8.4.

File 8D

Computer simulation exercise 8.4

Calculate the cut-off frequency of the circuit of Figure 8.11 if $R = 10\ \Omega$ and $L = 5$ mH. Simulate the circuit using these component values and perform an AC sweep to measure the response over a range from 1 Hz to 1 MHz. Plot the gain (in dB) and the phase of the output over this frequency range, estimate the cut-off frequency from these plots and compare this with the predicted value. Measure the phase shift at the estimated cut-off frequency and compare this with the value predicted above. Repeat this exercise for different values of R and L.

8.9 A comparison of *RC* and *RL* networks

From the above it is clear that *RC* and *RL* circuits have many similarities. The behaviour of the circuits we have considered is summarised in Figure 8.12. Each of the circuits has a cut-off frequency, and in each case this frequency is determined by the time constant T of the circuit. In the *RC* circuits $\mathrm{T} = CR$, and in the *RL* circuits $\mathrm{T} = L/R$. In each case the angular cut-off frequency is then given by $\omega_c = 1/\mathrm{T}$ and the cyclic cut-off frequency by $f_c = \omega_c/2\pi$.

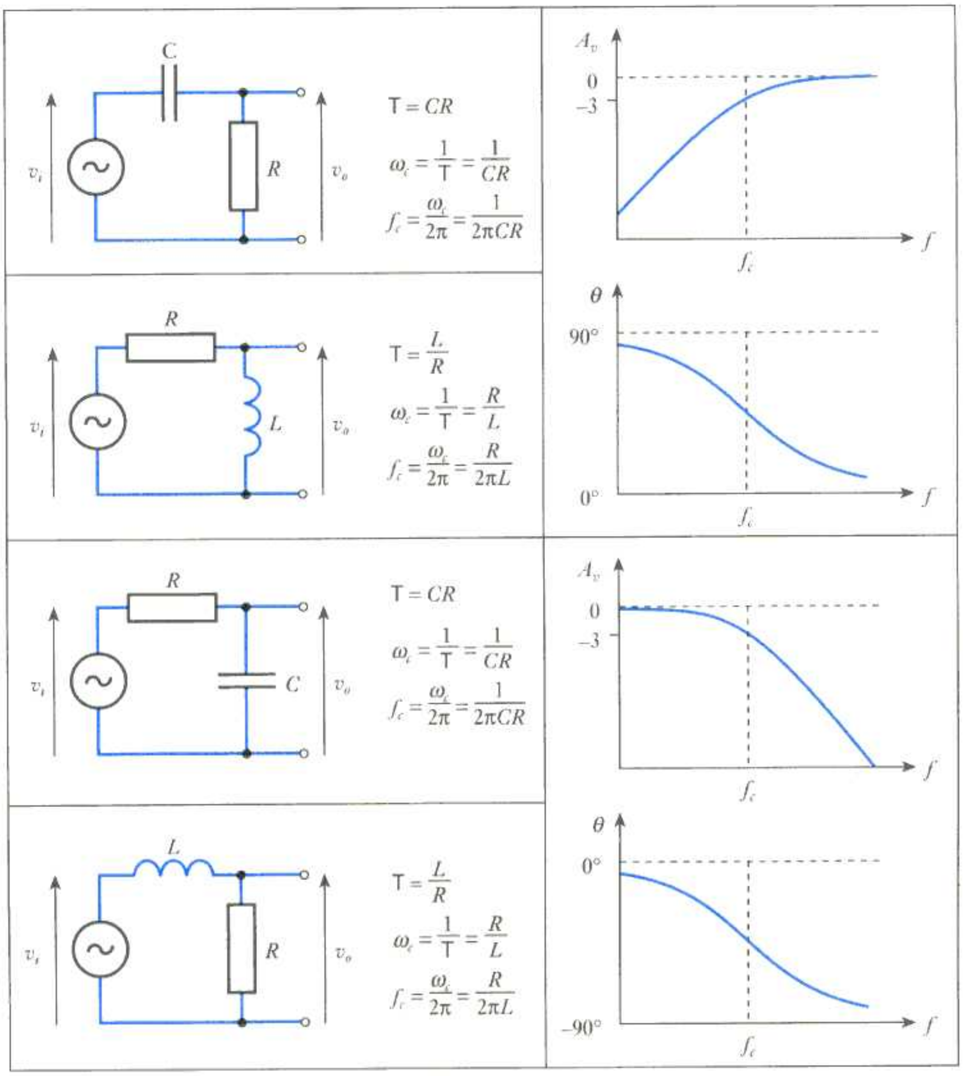


Figure 8.12 A comparison of *RC* and *RL* networks.

Two of the circuits of Figure 8.12 have high-frequency cut-offs (low-pass circuits), and two have low-frequency cut-offs (high-pass circuits). Transposing the components in a particular circuit will change it from a high-pass to a low-pass circuit, and vice versa. Replacing a capacitor by an inductor, or replacing an inductor by a capacitor, will also change it from a high-pass to a low-pass circuit, and vice versa.

File 8E

Computer simulation exercise 8.5

Calculate the time constants of the circuits of Figure 8.12 if $R = 1\ \text{k}\Omega$, $C = 1$ nF and $L = 1$ mH. Simulate the first of these circuits (using these component values) and use an AC sweep to plot the gain and phase responses of the circuit (as in the earlier simulation exercises in this chapter). Make a note of the cut-off frequency and confirm that this is a low-frequency cut-off. Now interchange the capacitor and resistor and again plot the circuit's characteristics. Note the effect on the cut-off frequency and the nature of the cut-off (that is, whether it is now a high- or a low-frequency cut-off).

Replace the capacitor by an inductor of 1 mH and again note the effect on the cut-off frequency and the nature of the cut-off. Finally, interchange the inductor and resistor and repeat the analysis. Hence confirm the form of the characteristics given in Figure 8.12.

8.10 Bode diagrams

Earlier we looked at Bode diagrams (also called Bode plots) as a means of describing the gain and phase response of a circuit (as in Figures 8.6 and 8.9). In the circuits we have considered, the gain at high and low frequencies has an asymptotic form, greatly simplifying the drawing of the diagram. The phase response is also straightforward, changing progressively between defined limits.

It is often sufficient to use a 'straight-line approximation' to the Bode diagram, simplifying its construction. For the circuits shown in Figure 8.12, we can construct the gain section of these diagrams simply by drawing the two asymptotes. One of these will be horizontal, representing the frequency range in which the gain is approximately constant. The other has a slope of +6 dB/octave (+20 dB/decade) or −6 dB/octave (−20 dB/decade), depending on whether this is a high-pass or low-pass circuit. These two asymptotes cross at the cut-off frequency of the circuit. The phase section of the response is often adequately represented by a straight-line transition between the two limiting values. The position of this line is defined by the phase shift at the cut-off frequency, which in these examples is 45°. A reasonable approximation to the response can be gained by drawing a straight line with a slope of −45°/decade through this point. Using this approach, the line starts one decade below the cut-off frequency and ends one decade above, making it very easy to construct. Straight-line approximations to the Bode diagrams for the circuits shown in Figure 8.12 are shown in Figure 8.13.

Once the straight-line Bode plots have been constructed, it is simple to convert these to a more accurate curved-line form if required. This can usually be done by eye by noting that the gain at the cut-off frequency is −3 dB, and that the phase response is slightly steeper at the cut-off frequency and slightly less steep near each end than the straight-line approximation. This is illustrated in Figure 8.14.

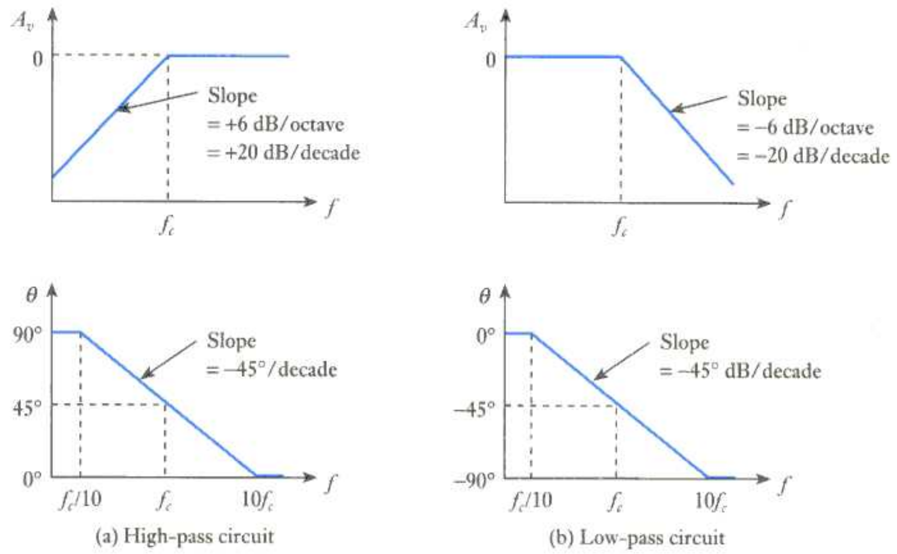
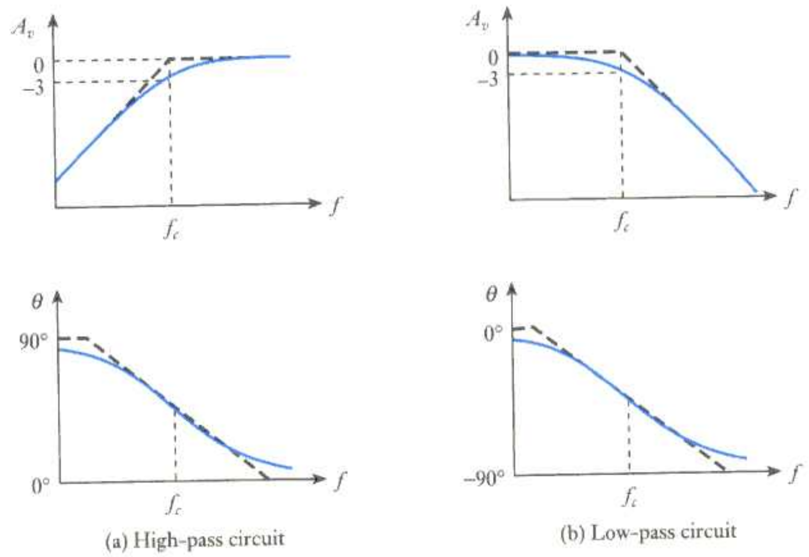


Figure 8.13 Simple straight-line Bode diagrams.

Figure 8.14 Drawing Bode diagrams from their straight-line approximations.

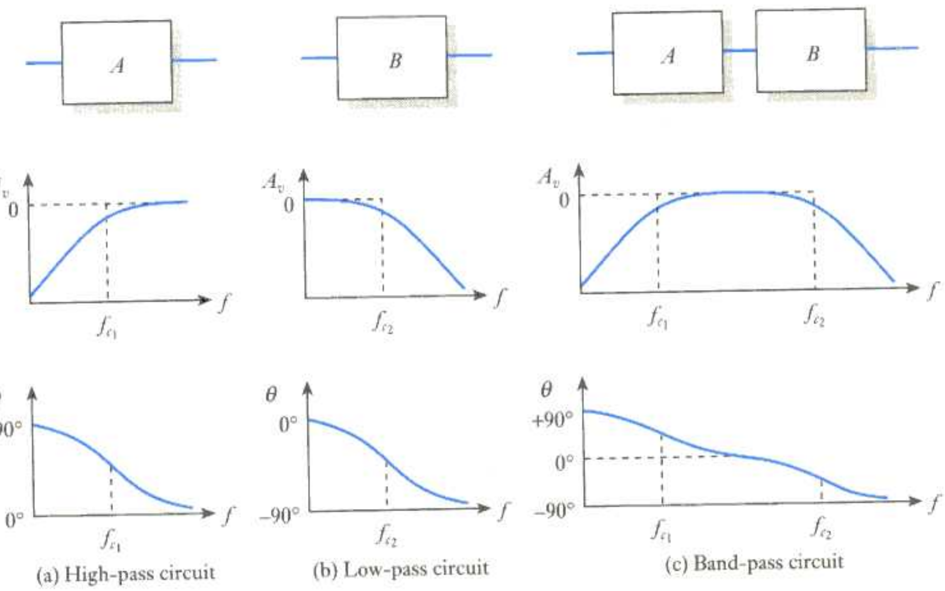


8.11 Combining the effects of several stages

While simple circuits may produce a single cut-off frequency, more complex circuits often possess a number of elements that each have some form of frequency dependence. Thus a circuit might have both high-pass and low-pass characteristics, or might have several high- or low-pass elements.

One of the advantages of the use of Bode diagrams is that they make it very easy to see the effects of combining several different elements. We noted in Section 8.3 that, when several stages of amplification are connected in series, the overall gain is equal to the product of their individual gains, or the *sum* of their gains when these are expressed in decibels. Similarly, the phase shift produced by several amplifiers in series is equal to the sum of the phase shifts produced by each amplifier separately. Therefore, the combined effects of a series of stages can be predicted by 'adding' the Bode diagrams of each stage. This is illustrated in Figure 8.15, which shows the effects of combining a high-pass and a low-pass element. In this case, the cut-off frequency of the high-pass element is lower than that of the low-pass element, resulting

Figure 8.15 The combined effects of high- and low-pass elements.



in a **band-pass filter** characteristic as shown in Figure 8.15(c). Such a circuit passes a given range of frequencies while rejecting lower- and high-frequency components.

Bode diagrams can also be used to investigate the effects of combining more than one high-pass or low-pass element. This is illustrated in Figure 8.16, which shows the effects of combining two elements that each contain a single high- and a single low-pass element. In this case, the cut-off frequencies of each element are different, resulting in four transitions in the characteristic. For obvious reasons, these frequencies are known as **break** or **corner frequencies**.

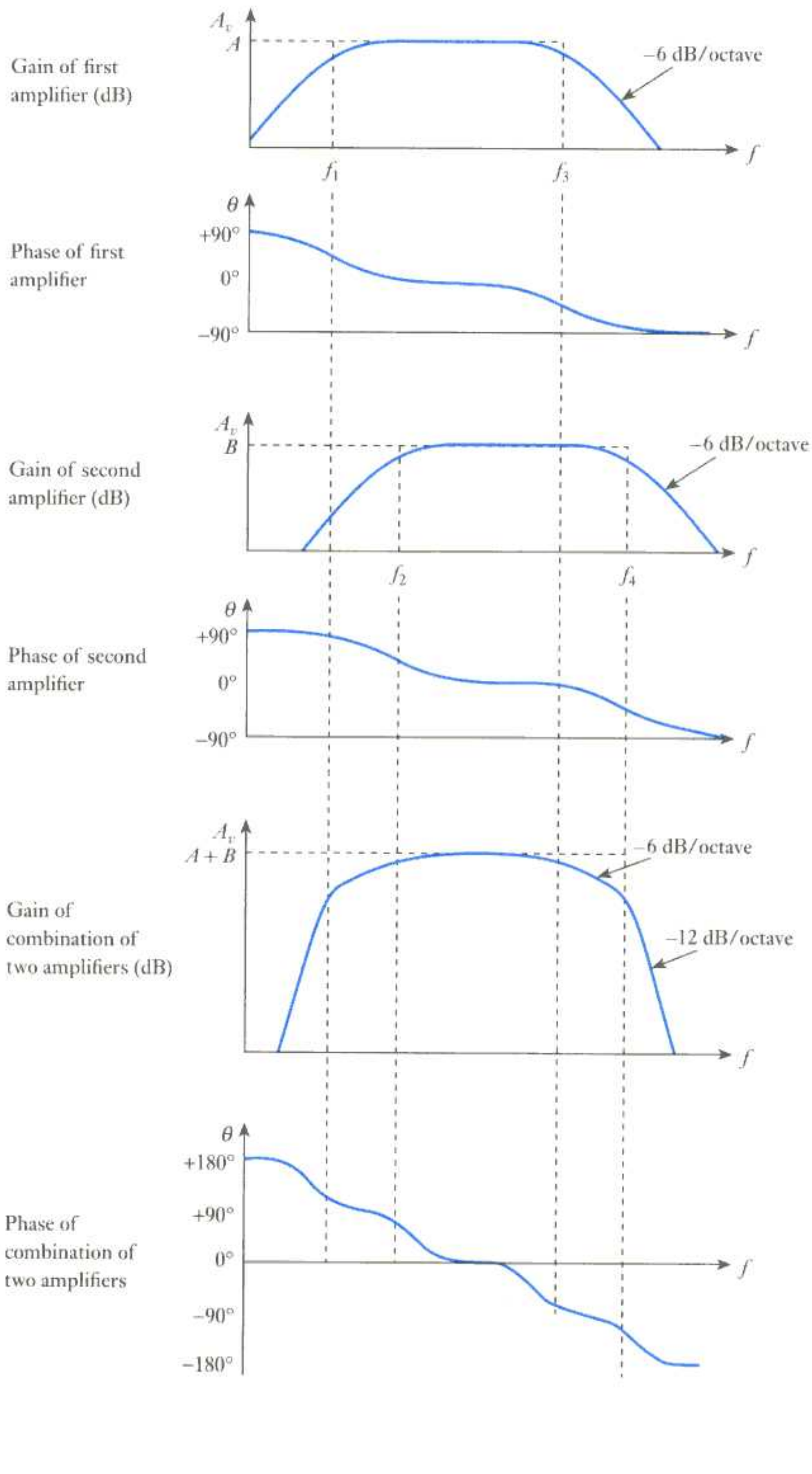


Figure 8.16 Combinations of multiple high- and low-pass elements.

In Figure 8.16, the first element is a band-pass amplifier that has a gain of A dB within its pass band, a low-frequency cut-off of f_1 and a high-frequency cut-off of f_3. The second element is also a band-pass amplifier, this time with a gain of B dB, a low-frequency cut-off of f_2 and a high-frequency cut-off of f_4. Within the frequency range from f_2 to f_3, the gains of both amplifiers are approximately constant, so the gain of the combination is also approximately constant, with a value of $(A + B)$ dB. In the range f_3 to f_4, the gain of the second amplifier is approximately constant, but the gain of the first falls at a rate of 6 dB/octave. Therefore, in this range the gain of the combination also falls at 6 dB/octave. At frequencies above f_4, the gain of both amplifiers is falling at a rate of 6 dB/octave, so the gain of the combination falls at 12 dB/octave. A similar combination of effects causes the gain to fall at first by 6 dB/octave, and then by 12 dB/octave as the frequency decreases below f_2. The result is a band-pass filter with a gain of $(A + B)$.

Within the pass band both amplifiers produce relatively little phase shift. However, as we move to frequencies above f_3 the first amplifier produces a phase shift that increases to −90°, and as we move above f_4 the second amplifier produces an additional shift, taking the total phase shift to −180°. This effect is mirrored at low frequencies, with the two amplifiers producing a total phase shift of +180° at very low frequencies.

While the arrangement represented in Figure 8.16 includes a total of two low-frequency and two high-frequency cut-offs, clearly more complex arrangements can include any number of cut-offs. As more are added, each introduces an additional 6 dB/octave to the maximum rate of increase or decrease of gain with frequency, and also increases the phase shift introduced at high or low frequencies by 90°.

8.12 *RLC* circuits and resonance

8.12.1 Series *RLC* circuit

Having looked at *RC* and *RL* circuits, it is now time to look at circuits containing resistance, inductance and capacitance. Consider for example the series arrangement of Figure 8.17. This can be analysed in a similar manner to the circuits discussed above, by considering it as a potential divider. The voltages across each component can be found by dividing its complex impedance by the total impedance of the circuit and multiplying this by the applied voltage. For example, the voltage across the resistor is given by

$$v_R = v \times \frac{\mathbf{Z}_R}{\mathbf{Z}_R + \mathbf{Z}_L + \mathbf{Z}_C} = v \times \frac{R}{R + j\omega L + \dfrac{1}{j\omega C}} \quad (8.21)$$

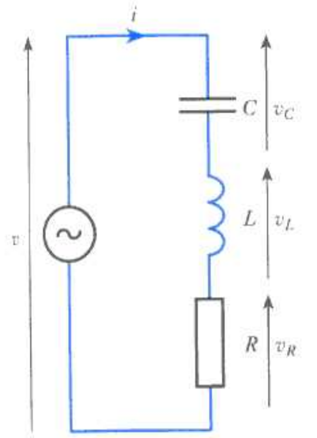


Figure 8.17 A series *RLC* arrangement.

It is also interesting to consider the impedance of this arrangement, which is given by

$$\mathbf{Z} = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right) \quad (8.22)$$

It can be seen that, if the magnitude of the reactance of the inductor and the capacitor are equal (that is, if $\omega L = 1/\omega C$), the imaginary part of the impedance is zero. Under these circumstances, the impedance of the arrangement is simply equal to R. This condition occurs when

$$\omega L = \frac{1}{\omega C} \qquad \omega^2 = \frac{1}{LC} \qquad \omega = \frac{1}{\sqrt{LC}}$$

This situation is referred to as **resonance**, and the frequency at which it occurs is called the **resonant frequency** of the circuit. An arrangement that exhibits such behaviour is known as a **resonant circuit**. The angular frequency at which resonance occurs is given the symbol ω_0, and the corresponding cyclic frequency is given the symbol f_0. Therefore

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{8.23}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \tag{8.24}$$

From Equation 8.22 it is clear that in the circuit of Figure 8.17 the impedance is at a *minimum* at resonance, and therefore the current will be at a *maximum* under these conditions. Figure 8.18 shows the current in the circuit as the frequency varies above and below resonance. Since the current is at a maximum at resonance, it follows that the voltages across the capacitor and the inductor are also large. Indeed, at resonance the voltages across these two components can be many times greater than the applied voltage. However, these two voltages are out of phase with each other and therefore cancel out, leaving only the voltage across the resistor.

We noted in Chapter 7 that power is not dissipated in capacitors or inductors but that these components simply store energy before returning it to the circuit. Therefore, the current flowing into and out of the inductor and capacitor at resonance results in energy being repeatedly stored and returned. This allows the resonant effect to be quantified by measuring the ratio of the energy stored to the energy dissipated during each cycle. This ratio is termed the **quality factor** or Q of the circuit. Since the energies stored in the inductor and the capacitor are equal, we can choose either of them to calculate Q. If we choose the inductor, we have

$$\text{quality factor } Q = \frac{I^2 X_L}{I^2 R} = \frac{X_L}{R} \tag{8.25}$$

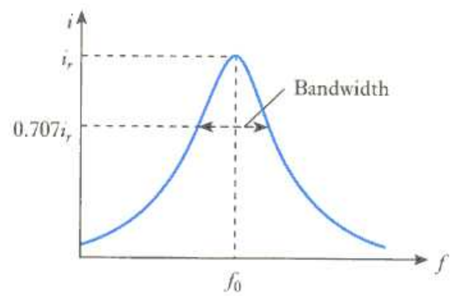


Figure 8.18 Variation of current with frequency for a series RLC arrangement.

and if we choose the capacitor, we have

$$\text{quality factor } Q = \frac{I^2 X_C}{I^2 R} = \frac{X_C}{R} \tag{8.26}$$

If we take either of these expressions and multiply top and bottom by I, we get the corresponding voltages across the associated component. Therefore, Q may also be defined as

$$\text{quality factor } Q = \frac{V_L}{V_R} = \frac{V_C}{V_R} \tag{8.27}$$

Since at resonance V_R is equal to the supply voltage, it follows that

$$\text{quality factor } Q = \frac{\text{voltage across } L \text{ or } C \text{ at resonance}}{\text{supply voltage}} \tag{8.28}$$

and thus Q represents the voltage magnification at resonance.

Combining Equations 8.23 and 8.28 gives us an expression for the Q of a series *RLC* circuit, which is

$$Q = \frac{1}{R}\sqrt{\left(\frac{L}{C}\right)} \tag{8.29}$$

The series *RLC* circuit is often referred to as an **acceptor circuit**, since it passes signals at frequencies close to its resonant frequency but rejects signals at other frequencies. We can define the bandwidth B of a resonant circuit as the frequency range between the points where the gain (or in this case the current) falls to $1/\sqrt{2}$ (or 0.707) times its mid-band value. This is illustrated in Figure 8.18. An example of an application of an acceptor circuit is in a radio, where we wish to accept the frequencies associated with a particular station while rejecting others. In such situations, we need a resonant circuit with an appropriate bandwidth to accept the wanted signal while rejecting unwanted signals and interference. The 'narrowness' of the bandwidth is determined by the Q of the circuit, and it can be shown that the resonant frequency and the bandwidth are related by the expression

$$\text{quality factor } Q = \frac{\text{resonant frequency}}{\text{bandwidth}} = \frac{f_0}{B} \tag{8.30}$$

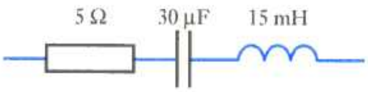
Combining Equations 8.24, 8.29 and 8.30, we can obtain an expression for the bandwidth of the circuit in terms of its component values. This is

$$B = \frac{R}{2\pi L} \text{ Hz} \tag{8.31}$$

It can be seen that reducing the value of R increases the Q of the circuit and reduces its bandwidth. In some situations it is desirable to have very high values of Q, and Equation 8.29 would suggest that if the resistor were omitted (effectively making $R = 0$) this would produce a resonant circuit with infinite Q. However, in practice all real components exhibit resistance (and inductors are particularly 'non-ideal' in this context), so the Q of such circuits is limited to a few hundred.

Example 8.9

For the following arrangement, calculate the resonant frequency f_0, the impedance of the circuit at this frequency, the quality factor Q of the circuit and its bandwidth B.



From Equation 8.24

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi\sqrt{15 \times 10^{-3} \times 30 \times 10^{-6}}}$$

$$= 237 \text{ Hz}$$

At the resonant frequency the impedance is equal to R, so $\mathbf{Z} = 5\ \Omega$.
From Equation 8.29

$$Q = \frac{1}{R}\sqrt{\left(\frac{L}{C}\right)} = \frac{1}{5}\sqrt{\left(\frac{15 \times 10^{-3}}{30 \times 10^{-6}}\right)} = 4.47$$

and from Equation 8.31

$$B = \frac{R}{2\pi L} = \frac{5}{2\pi \times 15 \times 10^{-3}} = 53 \text{ Hz}$$

File 8F

Computer simulation exercise 8.6

Simulate a circuit that applies a sinusoidal voltage to the arrangement of Example 8.9 and use an AC sweep to plot the variation of current with frequency. Measure the resonant frequency of the arrangement and its bandwidth, and hence calculate its Q. Measure the peak current in the circuit and, from a knowledge of the excitation voltage used, estimate the impedance of the circuit at resonance. Hence confirm the findings of Example 8.9 above.

8.12.2 Parallel *RLC* circuit

Consider now the parallel circuit of Figure 8.19. The impedance of this circuit is given by

$$\mathbf{Z} = \frac{1}{\frac{1}{R} + j\omega C + \frac{1}{j\omega L}} = \frac{1}{\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)} \quad (8.32)$$

and it is clear that this circuit also has a resonant characteristic. When $\omega C = 1/\omega L$, the term within the brackets is equal to zero and the imaginary part

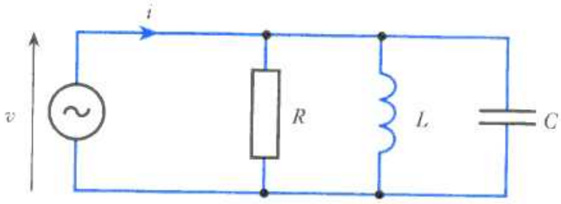


Figure 8.19 A parallel *RLC* arrangement.

of the impedance disappears. Under these circumstances, the impedance is purely resistive, and $\mathbf{Z} = R$. The frequency at which this occurs is the resonant frequency, which is given by

$$\omega C = \frac{1}{\omega L}$$

$$\omega^2 = \frac{1}{LC}$$

$$\omega = \frac{1}{\sqrt{LC}}$$

which is the same as for the series circuit. Therefore, as before, the resonant angular and cyclic frequencies are given by

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{8.33}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \tag{8.34}$$

From Equation 8.32, it is clear that the impedance of the parallel resonant circuit is a *maximum* at resonance and that it decreases at higher and lower frequencies. This arrangement is therefore a **rejector** circuit, and Figure 8.20 shows how the current varies with frequency.

As for the series resonant circuit, we can define both the bandwidth B and the quality factor Q for the parallel arrangement (although the definitions of these terms are a little different). The corresponding expressions for these quantities are

$$Q = R\sqrt{\left(\frac{C}{L}\right)} \tag{8.35}$$

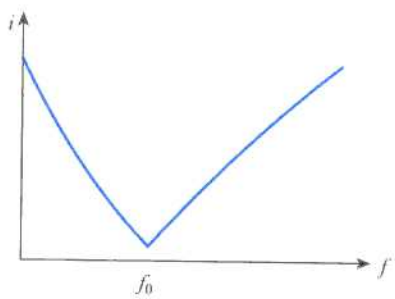
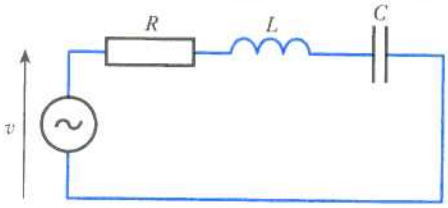
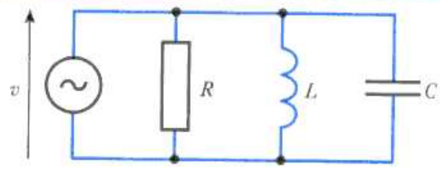


Figure 8.20 Variation of current with frequency for a parallel *RLC* arrangement.

Table 8.2 Series and parallel resonant circuits.

	Series resonant circuit	**Parallel resonant circuit**
Circuit		
Impedance, **Z**	$\mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)$	$\mathbf{Z} = \frac{1}{\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)}$
Resonant frequency, f_0	$f_0 = \frac{1}{2\pi\sqrt{LC}}$	$f_0 = \frac{1}{2\pi\sqrt{LC}}$
Quality factor, Q	$Q = \frac{1}{R}\sqrt{\left(\frac{L}{C}\right)}$	$Q = R\sqrt{\left(\frac{C}{L}\right)}$
Bandwidth, B	$B = \frac{R}{2\pi L}$ Hz	$B = \frac{1}{2\pi RC}$ Hz

and

$$B = \frac{1}{2\pi RC} \text{Hz} \tag{8.36}$$

A comparison between series and parallel resonant circuits is shown in Table 8.2. It should be noted that in a series resonant circuit Q is increased by *reducing* the value of R, while in a parallel resonant circuit Q is increased by *increasing* the value of R. In each case, Q is increased when the losses are reduced.

While the circuit of Figure 8.19 represents a generalised parallel RLC circuit, it is not the most common form. In practice, the objective is normally to maximise the Q of the arrangement, and this is achieved by removing the resistive element. However, in practice all inductors have appreciable resistance, so it is common to model this in the circuit as shown in Figure 8.21. Capacitors also exhibit resistance, but this is generally quite small and can often be ignored.

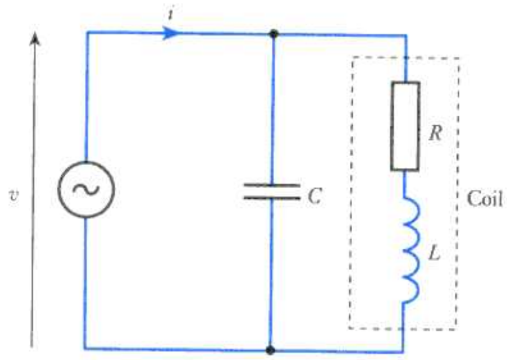


Figure 8.21 An LC resonant circuit.

The resonant frequency of the circuit of Figure 8.21 is given by

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \tag{8.37}$$

As the resistance of the coil tends to zero, this expression becomes equal to that of Equation 8.34. This circuit has similar characteristics to the earlier parallel arrangement and has a Q given by

$$Q = \sqrt{\frac{L}{R^2C} - 1} \tag{8.38}$$

8.13 Filters

8.13.1 *RC* filters

Earlier in this chapter we looked at *RC* high-pass and low-pass networks and noted that these have the characteristics of filters since they pass signals of certain frequencies while attenuating others. These simple circuits, which contain only a single time constant, are called **first-order** or **single-pole filters**. Circuits of this type are often used in systems to select or remove components of a signal. However, for many applications the relatively slow roll-off of the gain (6 dB/octave) is inadequate to remove unwanted signals effectively. In such cases, filters with more than one time constant are used to provide a more rapid roll-off of gain. Combining two high-pass time constants produces a second-order (two-pole) high-pass filter in which the gain will roll off at 12 dB/octave (as seen in Section 8.11). Similarly, the addition of three or four stages can produce a roll-off rate of 18 or 24 dB/octave.

In principle, any number of stages can be combined in this way to produce an nth-order (n-pole) filter. This will have a cut-off slope of $6n$ dB/octave and produce up to $n \times 90°$ of phase shift. It is also possible to combine high-pass and low-pass characteristics into a single band-pass filter if required.

For many applications, an **ideal filter** would have a constant gain and zero phase shift within one range of frequencies (its **pass band**) and zero gain outside this range (its **stop band**). The transition from the pass band to the stop band occurs at the **corner frequency** f_0. This is illustrated for a low-pass filter in Figure 8.22(a).

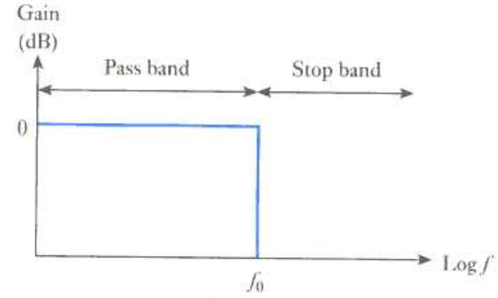
Unfortunately, although adding more stages to the *RC* filter increases the *ultimate* rate of fall of gain within the stop band, the sharpness of the 'knee' of the response is not improved (see Figure 8.22(b)). To produce a circuit that more closely approximates an ideal filter, different techniques are required.

8.13.2 *LC* filters

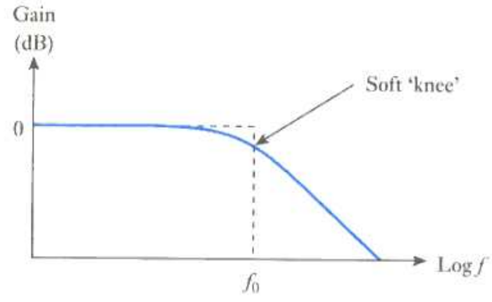
The combination of inductors and capacitors allows the production of filters with a very sharp cut-off. Simple *LC* filters can be produced using the series and parallel resonant circuits discussed in the last section. These are also known as **tuned circuits** and are illustrated in Figure 8.23.

These combinations of inductors and capacitors produce narrow-band filters with centre frequencies corresponding to the resonant frequency of the tuned circuit, so

Figure 8.22 Gain responses of ideal and real low-pass filters.

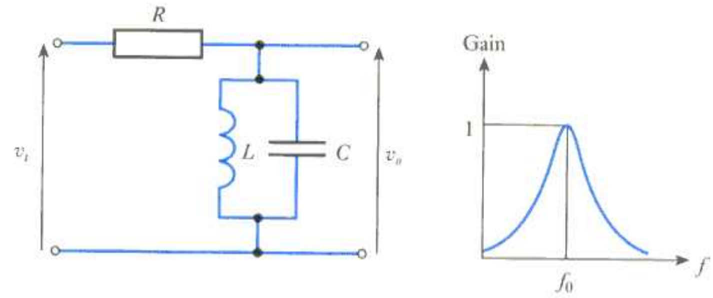


(a) An ideal low-pass filter

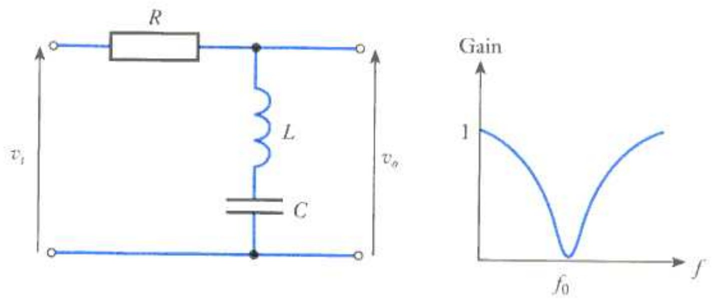


(b) A multi-stage *RC* filter

Figure 8.23 *LC* filters.



(a) A parallel *LC* network



(b) A series *LC* network

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \tag{8.39}$$

The bandwidth of the filters is determined by the **quality factor** Q as discussed in the last section.

Other configurations of inductors, capacitors and resistors can be used to form high-pass, low-pass, band-pass and band-stop filters and can achieve very high cut-off rates.

8.13.3 Active filters

Although combinations of inductors and capacitors can produce very high-performance filters, the use of inductors is inconvenient since they are expensive, bulky and suffer from greater losses than other passive components. Fortunately, a range of very effective filters can be constructed using an operational amplifier and suitable arrangements of resistors and capacitors. Such filters are called **active filters**, since they include an active component (the operational amplifier) in contrast to the other filters we have discussed, which are purely passive (ignoring any buffering). A detailed study of the operation and analysis of active filters is beyond the scope of this text, but it is worth looking at the characteristics of these circuits and comparing them with those of the *RC* filters discussed earlier.

To construct multiple-pole filters, it is often necessary to cascade many stages. If the time constants and the gains of each stage are varied in a defined manner, it is possible to create filters with a wide range of characteristics. Using these techniques, it is possible to construct filters of a number of different types to suit particular applications.

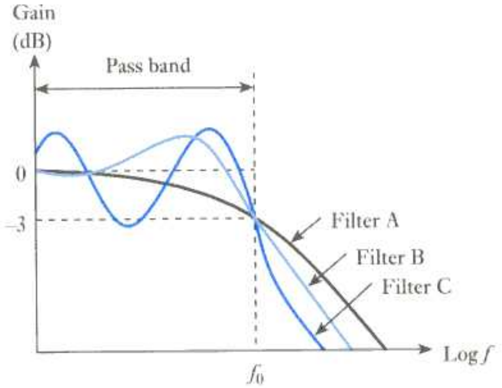
In simple *RC* filters, the gain starts to fall towards the edge of the pass band and so is not constant throughout the band. This is also true of active filters, but here the gain may actually rise towards the edge of the pass band before it begins to fall. In some circuits the gain fluctuates by small amounts right across the band. These characteristics are illustrated in Figure 8.24.

The ultimate rate of fall of gain with frequency for any form of active filter is $6n$ dB/octave, where n is the number of poles in the filter, which is often equal to the number of capacitors in the circuit. Thus the performance of the filter in this respect is related directly to circuit complexity.

Although the ultimate rate of fall of gain of a filter is defined by the number of poles, the sharpness of the 'knee' of the filter varies from one design to another. Filters with a very sharp knee tend to produce more variation in the gain of the filter within the pass band. This is illustrated in Figure 8.24, where it is apparent that filters B and C have a more rapid roll-off of gain than filter A but also have greater variation in their gain within the pass band.

Of great importance in some applications is the **phase response** of the filter: that is, the variation of phase lag or lead with frequency as a signal passes through the filter. We have seen that *RC* filters produce considerable amounts of phase shift within the pass band. All filters produce a phase shift

Figure 8.24 Variations of gain with frequency for various filters.



that varies with frequency. The way in which it is related to frequency varies from one type of filter to another. The phase response of a filter is of particular importance where pulses are to be used.

A wide range of filter designs are available, enabling one to be selected to favour any of the above characteristics. Unfortunately, the requirements of each are often mutually exclusive, so there is no universal optimum design and an appropriate circuit must be chosen for a given application. From the myriad of filter designs, three basic types are discussed here, first because they are widely used and second because they are each optimised for a particular characteristic.

The **Butterworth filter** is optimised to produce a flat response within its pass band, which it does at the expense of a less sharp 'knee' and a less than ideal phase performance. This filter is sometimes called a **maximally flat filter** as it produces the flattest response of any filter type.

The **Chebyshev filter** produces a sharp transition from the pass band to the stop band but does this by allowing variations in gain throughout the pass band. The gain ripples within specified limits, which can be selected according to the application. The phase response of the Chebyshev filter is poor, and it creates serious distortion of pulse waveforms.

The **Bessel filter** is optimised for a linear phase response and is sometimes called a **linear phase filter**. The 'knee' is much less sharp than for the Chebyshev or the Butterworth types (though slightly better than a simple *RC* filter), but its superior phase characteristics make it preferable in many applications, particularly where pulse waveforms are being used. The phase shift produced by the filter is approximately linearly related to the input frequency. The resultant phase shift therefore has the appearance of a fixed time delay, with all frequencies being delayed by the same time interval. The result is that complex waveforms that consist of many frequency components (such as pulse waveforms) are filtered without distorting the phase relationships between the various components of the signal. Each component is simply delayed by an equal time interval.

Figure 8.25 compares the characteristics of these three types of filter. Parts (a), (b) and (c) show the frequency responses for Butterworth, Chebyshev and Bessel filters, each with six poles (the Chebyshev is designed for 0.5 dB ripple), while (d), (e) and (f) show the responses of the same filters to a step input.

Over the years a number of designs have emerged to implement various forms of filter. The designs have different characteristics, and each has advantages and disadvantages. We will look at examples of active filter circuits in Chapter 15, when we look at operational amplifiers.

While active filters have several advantages over other forms of filter, it should be noted that they rely on the operational amplifier having sufficient gain at the frequencies being used. Active filters are widely used with audio signals (which are limited to a few tens of kilohertz) but are seldom used at very high frequencies. In contrast, *LC* filters can be used very successfully at frequencies up to several hundred megahertz. At very high frequencies, a range of other filter elements are available including SAW, ceramic and transmission line filters.

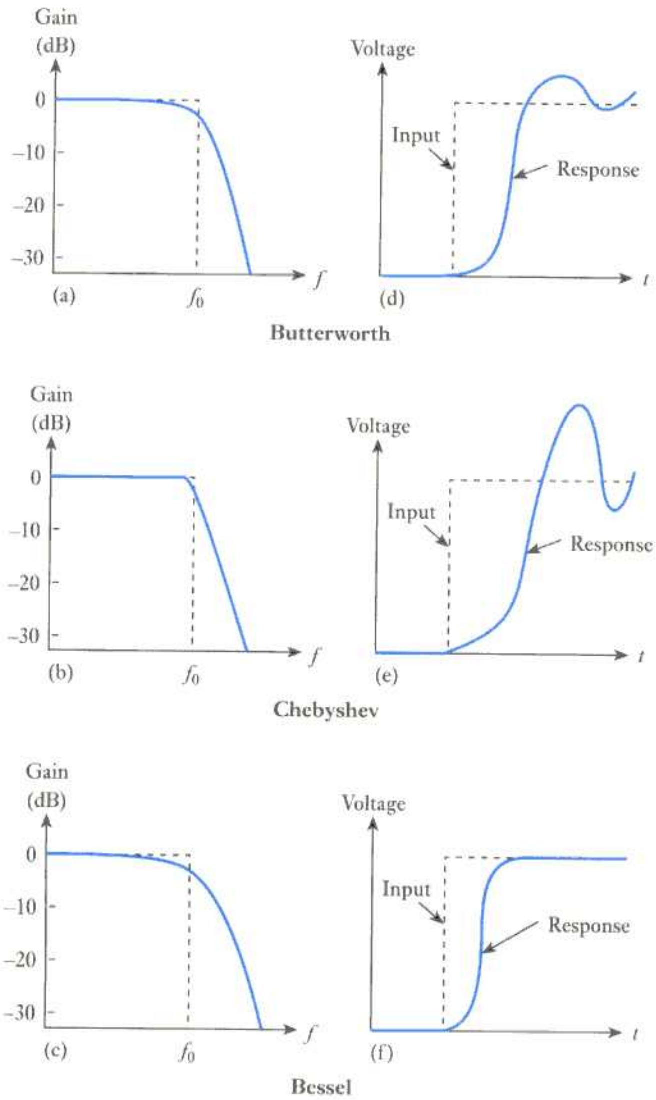


Figure 8.25 A comparison of Butterworth, Chebyshev and Bessel filters.

8.14 Stray capacitance and inductance

While many circuits will include a number of capacitors and inductors that have been intentionally introduced by the circuit designer, *all* circuits also include additional 'unintended' stray capacitances and stray inductances (as discussed in Chapters 4 and 5). Stray capacitance tends to introduce unintended low-pass filters in circuits, as illustrated in Figure 8.26(a). It also produces unwanted coupling of signals between circuits, resulting in a number of undesirable effects such as cross-talk. Stray inductance can also produce undesirable effects. For example, in Figure 8.26(b) a stray inductance L_s appears in series with a load resistor, producing an unintended low-pass effect. Stray effects also have a dramatic effect on the stability of circuits. This is illustrated in Figure 8.26(c), where stray capacitance C_s across an inductor L results in an unintended resonant circuit. We will return to look at stability in more detail in Chapter 22.

Stray capacitances and inductances are generally relatively small and therefore tend to be insignificant at low frequencies. However, at high frequencies they can have dramatic effects on the operation of circuits. In general, it is the presence of these unwanted circuit elements that limits the high-frequency performance of circuits.

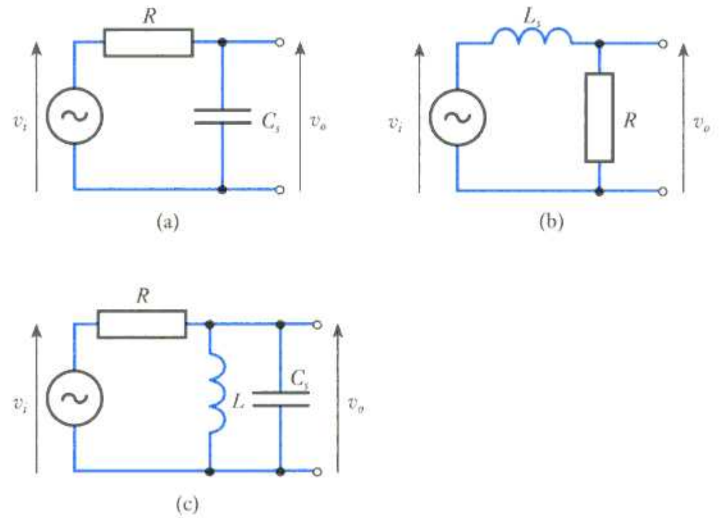


Figure 8.26 The effects of stray capacitance and inductance.

Key points

- The reactance of capacitors and inductors is dependent on frequency. Therefore, the behaviour of any circuit that contains these components will change with frequency.
- Since all real circuits include stray capacitance and stray inductance, all real circuits have characteristics that change with frequency.
- Combinations of a single resistor and a single capacitor, or a single resistor and a single inductor, can produce circuits with a single high- or low-frequency cut-off. In each case, the angular cut-off frequency ω_c is given by the reciprocal of the time constant T of the circuit.
- For an RC circuit $T = CR$, while in an RL circuit $T = L/R$.
- These single time constant circuits have certain similar characteristics:
  - their cut-off frequency $f_c = \omega_c/2\pi = 1/2\pi T$;
  - at frequencies well away from their cut-off frequency within their pass band, they have a gain of 0 dB and zero phase shift;
  - at their cut-off frequency, they have a gain of −3 dB and ±45° phase shift;
  - at frequencies well away from their cut-off frequency within their stop band, their gain changes by ±6 dB/octave (±20 dB/decade) and they have a phase shift of ±90°.
- Gain and phase responses are often given in the form of a Bode diagram, which plots gain (in dB) and phase against log frequency.
- When several stages are used in series, the gain of the combination at a given frequency is found by multiplying their individual gains, while the phase shift is found by adding their individual phase shifts.
- Combinations of resistors, inductors and capacitors can be analysed using the tools covered in earlier chapters. Of particular interest is the condition of resonance, when the reactance of the capacitive and inductive elements cancels. Under these conditions, the impedance of the circuit is simply resistive.

- The 'sharpness' of the resonance is measured by the quality factor Q.
- Simple RC and RL circuits represent first-order, or single-pole, filters. Although these are useful in certain applications, they have a limited 'roll-off' rate and a soft 'knee'.
- Combining several stages of RC filters increases the roll-off rate but does not improve the sharpness of the knee. Higher performance can be achieved using LC filters, but inductors are large, heavy and have high losses.
- Active filters produce high performance without using inductors. Several forms are available to suit a range of applications.
- Stray capacitance and stray inductance limit the performance of all high-frequency circuits.

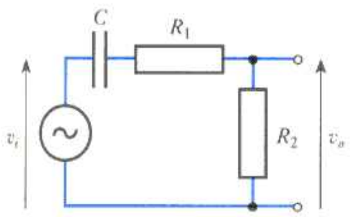
Exercises

8.1 Calculate the reactance of a 1 μF capacitor at a frequency of 10 kHz, and the reactance of a 20 mH inductor at a frequency of 100 rad/s. In each case include the units in your answer.

8.2 Express an angular frequency of 250 rad/s as a cyclic frequency (in Hz).

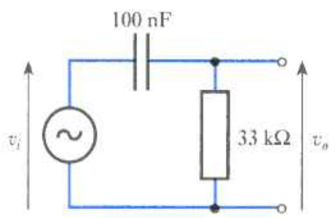
8.3 Express a cyclic frequency of 250 Hz as an angular frequency (in rad/s).

8.4 Determine the transfer function of the following circuit.



8.5 A series RC circuit is formed from a resistor of 33 kΩ and a capacitor or 15 nF. What is the time constant of this circuit?

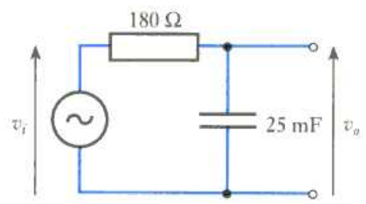
8.6 Calculate the time constant T, the angular cut-off frequency ω_c and the cyclic cut-off frequency f_c of the following arrangement. Is this a high- or a low-frequency cut-off?



8.7 Simulate the arrangement of Exercise 8.6 and use an AC sweep to display the gain response. Measure the cut-off frequency of the circuit and hence confirm your results for the previous exercise.

8.8 Determine the frequencies that correspond to:
(a) an octave below 30 Hz;
(b) two octaves above 25 kHz;
(c) three octaves above 1 kHz;
(d) a decade above 1 MHz;
(e) two decades below 300 Hz;
(f) three decades above 50 Hz.

8.9 Calculate the time constant T, the angular cut-off frequency ω_c and the cyclic cut-off frequency f_c of the following arrangement. Is this a high- or a low-frequency cut-off?

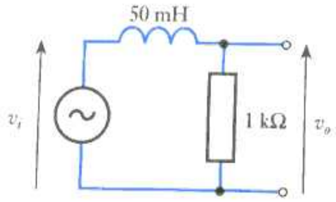


8.10 Simulate the arrangement of Exercise 8.9 and use an AC sweep to display the gain response. Measure the cut-off frequency of the circuit and hence confirm your results for the previous exercise.

8.11 A parallel RL circuit is formed from a resistor of 150 Ω and an inductor of 30 mH. What is the time constant of this circuit?

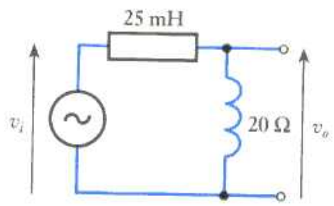
8.12 Calculate the time constant T, the angular cut-off frequency ω_c and the cyclic cut-off frequency f_c of the following arrangement. Is this a high- or a low-frequency cut-off?

Exercises continued



8.13 Simulate the arrangement of Exercise 8.12 and use an AC sweep to display the gain response. Measure the cut-off frequency of the circuit and hence confirm your results for the previous exercise.

8.14 Calculate the time constant T, the angular cut-off frequency ω_c and the cyclic cut-off frequency f_c of the following arrangement. Is this a high- or a low-frequency cut-off?



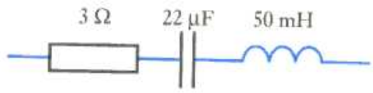
8.15 Simulate the arrangement of Exercise 8.14 and use an AC sweep to display the gain response. Measure the cut-off frequency of the circuit and hence confirm your results for the previous exercise.

8.16 Sketch a straight-line approximation to the Bode diagram of the circuit of Exercise 8.14. Use this approximation to produce a more realistic plot of the gain and phase responses of the circuit.

8.17 A circuit contains three high-frequency cut-offs and two low-frequency cut-offs. What are the rates of change of gain of this circuit at very high and very low frequencies?

8.18 Explain what is meant by the term 'resonance'.

8.19 Calculate the resonant frequency f_0, the quality factor Q and the bandwidth B of the following circuit.



8.20 Simulate a circuit that applies a sinusoidal voltage to the arrangement of Exercise 8.19 and use an AC sweep to plot the variation of current with frequency. Measure the resonant frequency of the arrangement and its bandwidth, and hence calculate its Q. Hence confirm your results for the previous exercise.

8.21 Explain the difference between a passive and an active filter.

8.22 Why are inductors often avoided in the construction of filters?

8.23 What form of active filter is optimised to produce a flat response within its pass band?

8.24 What form of active filter is optimised to produce a sharp transition from the pass band to the stop band?

8.25 What form of filter is optimised for a linear phase response?

8.26 Explain why stray capacitance and stray inductance affect the frequency response of electronic circuits.