HÁSKÓLI ÍSLANDS **Eðlisfræðiskor**
Raunvísindadeild **Eðlisfræði I**

Pendulum

References

1. Young & Freedman: *University Physics*, kafli 13.5.
2. Benson: *University Physics*, kaflar 15.1 og 15.4

1 Introduction

In this lab exercise the student will get to know a simple model, describing the period of a simple pendulum, that breaks down when carefully scrutinized. We shall explore when deviations from the simple model are observable and the model will be improved in an attempt to better represent reality. This lab is divided in two parts, the first where the period is measured with a stopwatch and the second where it is measured with an electronic clock connected to an optical gate.

2 The simple pendulum

The simple pendulum is described in the textbood (see reference). Forces parallel to the direction of motion in Fig. 13.20 in Young & Freedman are equal to ma. Rearrangement of this statement results in the equation of motion of the simple pendulum

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\sin(\theta(t)) = 0 \tag{1}$$

It is worth noting that m does not appear in Eq.(1). The motion of the pendulum is thus independent of its mass. For small angles θ we can approximate $\sin(\theta)$ by θ, or $\sin(\theta) \cong \theta$. Equation (1) then has a simple solution

$$\theta(t) = \Theta\cos(\omega t) \tag{2}$$

where Θ is the maximum amplitude of the pendulum and $\omega^2 = g/L$. The period, $T_0 = \frac{2\pi}{\omega}$, is thus

$$T_0 = 2\pi\sqrt{\frac{L}{g}} \tag{3}$$

The importance of the pendulum lies among other things in that its period is independent of the mass of the weight and the oscillation amplitude within the limits of our approximation earlier. As the amplitude is increased the approximation $\sin(\theta) \cong \theta$ breaks down. When this becomes observable [as deviation from Eq.(3)] depends on the accuracy of the time measurement used to determine the period. When this happens we do what physicists usually try to do, we improve our model.

3 Improved model

Equation (1) cannot be solved directly without approximation to obtain a closed form solution as in Eq.(2). However it is possible to obtain the period of oscillation. This will not be done here, but with the appropriate method[1] one obtains the period

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\Theta} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\Theta)}} \tag{4}$$

This integral is solved in the *appendix* and the solution is represented as an infinite series. Here we display the first three terms

$$T = T_0\left[1 + \frac{1}{4}\sin^2\frac{\Theta}{2} + \frac{9}{64}\sin^4\frac{\Theta}{2} + \dots\right] \tag{5}$$

To explore how many terms are needed for a useful description of the motion we insert $\Theta = 60°$. The we obtain $T = T_0(1 + 1/16 + 9/1024 + \dots)$, so the third term is less than 1% of the first term. Let us assume that the error in using only two terms is less that the error of the time measurement and ignore the third term, for now. Now we have simplified our improved model to

$$T_\Theta = T_0\left(1 + \frac{1}{4}\sin^2\frac{\Theta}{2}\right) \tag{6}$$

The first term in Eq.(6) is the classical model (Eq.(3)) and the second one describes how the amplitude Θ affects the period T_Θ.

4 Measurements of period of pendulum

The pendulum is a weight that dangles from a gallow with a soft, lightweight thread. On the gallows there's a protractor to allow measurements of amplitude of oscillations.

- Measure the pendulum's length, L. Note that the weight is not a point mass as in Fig. 13.20 in Young & Freedman. You must therefore define the length L sensibly.

4.1 Measurements with a stopwatch

- Measure the period of oscillation using a stopwatch for $\Theta < 5°$, $\Theta = 15°$ and $\Theta = 30°$. Consider whether it makes sense to measure a single period or to count several/many and devide by the number of oscillations. Why not time 100 periods? Discuss the pros and cons amongst yourselves.

- Compare the measurement results (Θ, T_Θ) with the model. Remember to take uncertainty/error into account.

Suggestion for data processing: To find the uncertainty in the model value of the period T_0 we differentiate the model (3) (see e.g. græna kver or notes on the webpage)

$$\Delta T_0 = \frac{\partial T_0}{\partial L}\cdot \Delta L$$

and obtain

$$\Delta T_0 = 2\pi \cdot \frac{\partial}{\partial L}\sqrt{\frac{L}{g}}\cdot \Delta L$$

[1]Ragnar Sigurðsson, *Fyrirlestrar um afleiðujöfnur, Fourier-greiningu og tvinnfallagreiningu,* Háskólaútgáfan, 1994, bls. 9 - 10

or

$$\Delta T_0 = \frac{\pi \cdot \Delta L}{\sqrt{g \cdot L}}$$

To connect the uncertainty in an n-period measurement T_n to the uncertainty for a single period we define the quantity $\overline{T} = \frac{T_n}{n}$. We then have the relation

$$\Delta \overline{T} = \frac{1}{n} \cdot \Delta T_n$$

Check if the model and your measurements agree within uncertainty.

4.2 Measurements with an electronic clock

- Set the clock to *Period Mode* and measure the period for amplitudes Θ in the range 5° - 40° at approximately 5° intervals. *The clock's manual is in an appendix.*
- Compare the measurements (Θ, T_Θ) with the model. Remember to take into account uncertainty.

Suggested data processing: The model describes the relation between Θ and T_Θ, that is *non-linear*. This makes graphical comparison difficult. What can one do? It is probably easiest to rewrite Eq.(6) so there is a linear relationship between two variables. A very simple manipulation of (6) yields

$$T_\Theta = T_0 + \frac{T_0}{4}\left(\sin^2 \frac{\Theta}{2}\right) \tag{7}$$

Now there's a linear relationship between T_Θ and the quantity $\sin^2 \frac{\Theta}{2}$. Make a graph with $\sin^2 \frac{\Theta}{2}$ on the horizontal axis and T_Θ on the vertical axis. The data should reveal a linear relationship where the intercept with T_Θ is at T_0 and the slope of the line is $\frac{T_0}{4}$.

The uncertainty in $\sin^2 \frac{\Theta}{2}$ can be obtained by

$$\Delta \sin^2 \frac{\Theta}{2} = \frac{\partial \left(\sin^2 \frac{\Theta}{2}\right)}{\partial \Theta} \Delta \Theta = \frac{1}{2} \sin \Theta \Delta \Theta$$

Note that the quantity $\Delta\Theta$ must be in radians because our rules of differentiation to obtain uncertainty only apply in those units of measure (od not use degrees for angles when dealing with uncertainty).

The stability of the setup may have a greater effect on the uncertainty in period than the properties of the clock. The uncertainty ΔT_Θ must cover all variations in measurements for a fixed amplitude.